Fuzzy Modelling for Uncertainty Propagation and Risk Quantification in Environmental Water Systems

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Abstract. Risk-based management of environmental systems, like rivers, lakes, groundwater aquifers and coastal areas, is a very useful approach for combating specific problems, like water pollution and loss of ecosystems biodiversity. Uncertainties that could be inherent to natural variabilities in space and time, such as those due to hydrological and climatic variations, together with uncertainties related to human activities or terrorist attacks, may produce various risks and failures affecting both human health and ecosystems integrity. The fuzzy set theory, in combination with mathematical modelling based on partial differential equations, is proposed in this paper, in order to propagate uncertainties in estimating output variables in water quality problems of water systems. Uncertainties in input variables and values of model parameters are first introduced as fuzzy numbers. Then, they are propagated using fuzzy arithmetic. Output variables, like water pollution and environmental risk, are estimated in terms of fuzzy numbers.

Keywords: Environmental security; pollution; risk; uncertainty, fuzzy numbers

Introduction

Environmental systems like coastal areas, lakes and rivers are complex, not only because of their heterogeneity and huge spatial and temporal variations of their physical, chemical and biological properties, but also because of human interventions. Release of pollutants from various sources, like domestic, industrial and agricultural water uses, may affect the quality properties of natural water and deteriorate the functioning of various ecosystems. Intentional damage to the quality of environmental water in such systems from terrorist attacks may also become a major threat to human health and environmental security.

In order to take into account major uncertainties for modelling the behaviour of such systems, stochastic modelling has been applied in the past [1], [2], [3]. According to stochastic simulation, physical parameters and input loads are considered as *random variables*. This is a frequency-based approach to propagate uncertainties. Results of probability theory (stochastic arithmetic) have been introduced in stochastic modelling

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in the form of analytical functional relationships between random variables, or by simulating a large number of different realisations (Monte Carlo method) [3].

In this paper, the fuzzy set theory in combination with mathematical modelling is proposed in order to assess uncertainties in estimating environmental water pollution. First, uncertainties in input loads and values of physical parameters are introduced as *fuzzy numbers*, and then uncertainties are propagated by using fuzzy calculus. With fuzzy mathematical modelling it is possible to assimilate imprecise data and directly produce imprecise output in the form of fuzzy numbers without repeating a large number of computations.

Examples of application in simplified and complicated real cases illustrate the capabilities of the above methodology and the precautions needed to be taken for its successful implementation in water pollution problems.

1. Types of Uncertainty

In managing complex environmental systems, there are several types of uncertainties and risks. These are caused by different uncertainties and imprecision, such as the high variability in space and time of the hydrodynamic, chemical and biological processes involved and also man-induced uncertainties.

Uncertainties are due to lack of knowledge about the structure of various physical and biochemical processes and also to the limited amount of data available [2], [7]. Several authors in the literature have analysed different types of uncertainties and made various distinctions, such as between objective and subjective, basic and secondary, natural and technological uncertainties.

Distinction should be made between

(1) aleatory or natural uncertainties or randomness, and
(2) epistemic or man-induced or technological uncertainties.

1.1. Aleatory Uncertainties or Randomness

It is postulated that natural uncertainties are inherent to the specific process and that they cannot be reduced by use of an improved method or more sophisticated models. They are linked to natural variability both in space and time. Uncertainties due to natural randomness or aleatory uncertainties may be taken into account by using the stochastic or fuzzy approaches.

1.2. Epistemic or Man-induced Uncertainties

Man-induced uncertainties are of different kinds: (a) data uncertainties, due to sampling methods (statistical characteristics), measurement errors and methods of analysing the data (b) modelling uncertainties, due to the inadequate mathematical models in use and to errors in parameter estimation, and (c) operational uncertainties, which are related generally to the construction, maintenance and operation of engineering works. Contrary to natural randomness, man-induced uncertainties may be reduced by collecting more information or by improving the mathematical model. As can be seen later on in this paper, in a Bayesian framework prior information may be increased into posterior information, by use of additional information or data.

Alternatively, when data are scarce, the fuzzy set theory may be used to handle and quantify imprecision.

The fate of pollutants in a water receiving body, such as a river, is influenced by the combination of three mechanisms: (a) advection by currents (b) turbulent diffusion and (c) chemical, biological or other interactions. As a result, collected data of physical and chemical parameters are very irregular in time, as shown in Figure 1 for typical time series of nitrate concentration.

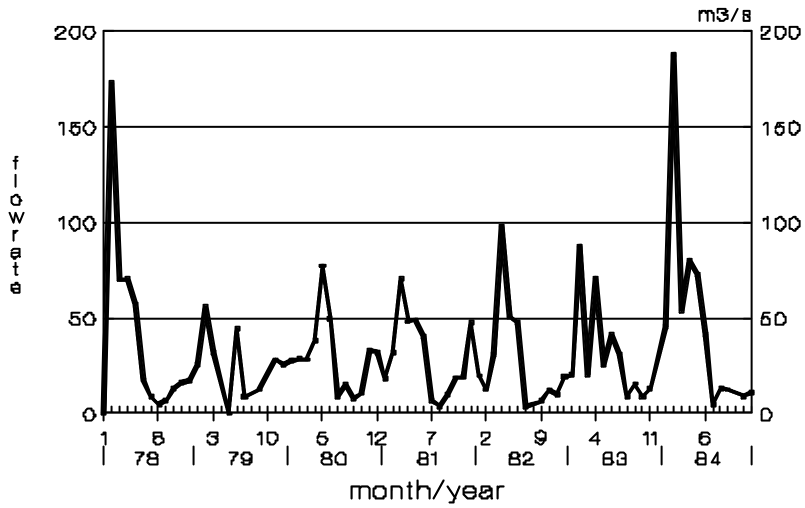


Figure 1. Uncertainties in time series of nitrate concentration

2. Probabilistic and Fuzzy Definitions of Risk

As explained in [7], the load ℓ may be defined as a variable reflecting certain external conditions under which the system may be stressed. There is a characteristic variable describing the capacity of the system to overcome this external load. This system variable may be defined as the resistance r.

In order to define the probabilistic risk, one should answer three main questions:

1. When might the system fail?
2. How often might the system fail? and
3. What are the consequences when there is a failure?

In general, different measures of risk may be defined. The more general is to consider all three of the above questions, i.e. the scenario (critical condition), the frequency or probability of failure and the consequences. An incident or even failure will occur when the load exceeds this resistance, i.e., FAILURE or INCIDENT : $\ell > r$

SAFETY or RELIABILITY : $\ell \leq r$

The product of probability of failure multiplied by the consequences may be taken as a measure of risk, representing the expected or probable consequences. More generally, risk may be defined as a function of two variables: (1) the probability of failure and (2) the consequences.

Engineers have often considered a measure of risk as just being the probability of failure [3]. In a probabilistic framework, if ℓ and r are considered as random or stochastic variables the result is:

ENGINEERING RISK= probability of failure= $P(\ell > r)$

ENGINEERING RELIABILITY= probability of success= $P(\ell \leq r)$

As shown in Figure 2, if $f_{LR}(\ell, r)$ is the joint probability density function, the risk p_F may be estimated by integrating the function $f_{LR}(\ell, r)$ above the bisectrice line L=R. For calculating the engineering risk, the following formula is obtained:

$$p_F = P(L > R) = \int_0^{\infty} \left(\int_0^{\ell} f_{LR}(\ell, r)\,dr\right) d\ell \qquad (1)$$

This is a general expression for quantifying risk in a probabilistic framework. However, Equation 1 is rather difficult to use, because most of the time the joint density probability function $f_{LR}(\ell, r)$ is unknown. Simplifications include the assumption of independence between load and resistance, or the case when one of the two variables is deterministic.

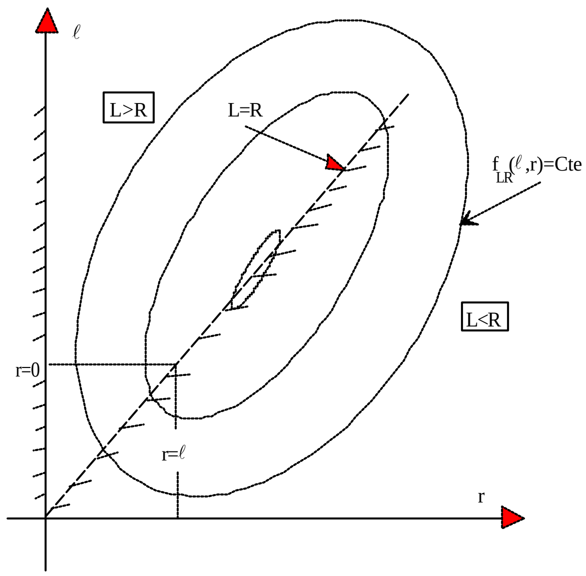


Figure 2. Calculation of the engineering risk by integrating the joint density probability function $f_{LR}(\ell, r)$

If the resistance r is taken as deterministic with a constant value of r=R, then Equation 1 is written as follows:

$$p_F = P(L > R) = \int_R^{\infty} f_L(\ell)d\ell \tag{2}$$

If fuzzy logic is used, ℓ and r are considered as fuzzy numbers, noted as $\tilde{L}$ and $\tilde{R}$ (see Appendix). Then risk and reliability are defined by means of appropriate fuzzy measures, which are introduced below.

Consider that the system has a resistance $\tilde{R}$ and a load $\tilde{L}$, both represented by fuzzy numbers. A *reliability measure* or a *safety margin* of the system may be defined as being the difference between load and resistance [20], [7]. This is also a fuzzy number given by

$$\tilde{M} = \tilde{R} - \tilde{L}$$

Taking the h-level intervals of $\tilde{R}$ and $\tilde{L}$ as

$$R(h)=[R_1(h), R_2(h)],$$
$$L(h)=[L_1(h), L_2(h)],$$

then, for every $h \in [0, 1]$, the *safety margin* M(h) is obtained by subtracting L(h) from R(h), i.e.

$$M(h) = R(h) - L(h) .$$

Two limiting cases may be distinguished, as shown in Figure 3:
There is *absolute safety* if:

$$M(h) \geq 0 \qquad \forall\ h\ [0,1]$$

whereas *absolute failure* occurs when:

$$M(h) < 0 \qquad \forall\ h\ [0,1] .$$

A *fuzzy measure of risk*, or *fuzzy risk index R_i* may be defined as the area of the fuzzy safety margin, *where values of M are negative*. Mathematically, this may be shown as:

$$R_i = \frac{\int_{m<0} \mu_{\tilde{M}}(m)dm}{\int_m \mu_{\tilde{M}}(m)dm}$$

The fuzzy measure of reliability, or fuzzy reliability index Re is the complement of Ri, i.e.

$$R_e = 1 - R_i = \frac{\int_{m>0} \mu_{\tilde{M}}(m)dm}{\int_m \mu_{\tilde{M}}(m)dm}$$

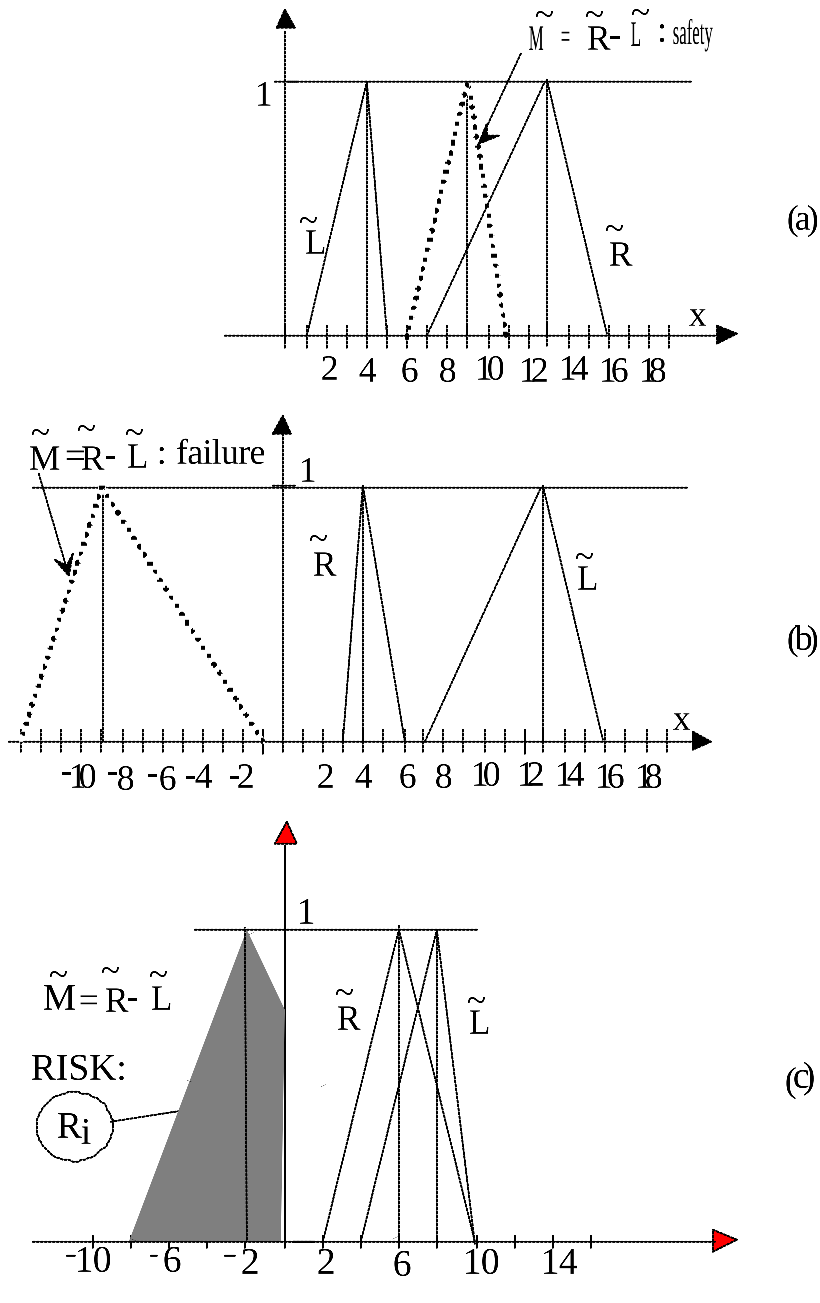


Figure 3. Absolute safety (a), absolute failure (b) and fuzzy risk (c).

3. Stochastic Modelling

Deterministic modelling introduces sharp values of problem variables, model parameters and boundary conditions. If the mathematical model consists of a set of partial differential equations, then analytical or numerical solutions may be found by use of finite differences or finite elements. This approach is not adequate to incorporate imprecision on data and model parameters, propagate uncertainties and proceed with risk and reliability analysis.

According to stochastic simulation, problem variables, like hydrological, hydrodynamic and water quality, model parameters and input loads are considered as *random variables*. This is a frequency-based approach, which is able to propagate uncertainties. Results of probability theory (stochastic arithmetic) may be introduced in stochastic modelling in the form of analytical functional relationships between random variables, or by simulating a large number of different realisations (Monte Carlo method). In stochastic modelling risk and reliability analysis of water quantity and quality may be also considered.

The connection between deterministic and stochastic approaches has been analysed for the case of aquifer systems [2]. Furthermore, various methods and tools have been developed in the past for stochastic simulation and risk quantification, such as: (1) Time series analysis, filtering, krigging, (2) Stochastic differential equations, (3) Spectral analysis, (4) Perturbation analysis, and (5) Monte-Carlo simulation.

More recently, the United States Environmental Protection Agency has developed guidelines for the application of more advanced methodologies for risk assessment and management [24], [25], [26]. *Ecological Risk Assessment* (ERA) evaluates the likelihood that adverse ecological effects may occur or are occurring as a result of exposure to one or more stressors. The *Cumulative Risk Assessment* methodology was developed to assess risks and make environmental protection decisions based not only on individual contaminants, such as lead, chlordane, and DDT, but by describing and quantifying the risks from many sources of pollution. *Comparative Risk Assessment* is a framework that uses sound scientific, economic and policy analysis as well as stakeholder participation to identify and compare the areas at greatest environmental risk and provide a methodology for prioritising environmental problems.

Stochastic modelling is considered in this paper in connection with pollution of natural water systems. Deterioration of groundwater and surface water quality from natural and anthropogenic causes is of growing concern in many parts of the world. Increased water pollution, caused by urban, industrial and agricultural activities, is directly related to the constant growth in the number of people living in river basins and coastal areas. Huge amounts of nitrogen and phosphorus are released from sewage and leaching of fertilisers resulting in algal blooming, eutrophication, shellfish die-off and loss of habitat, together with severe economic, social and ecological losses. The situation becomes critical in lakes and semi-enclosed seas such as the Mediterranean, where water exchange with the outer ocean is quite small.

Simulation of environmental water quality is based on the well-known advection-dispersion mathematical model. The rate of change of the concentration of n different pollutants under biochemical interactions can be expressed in two dimensions as:

$$\frac{\partial c_k}{\partial t} + u\frac{\partial c_k}{\partial x} + v\frac{\partial c_k}{\partial y} = D_x \frac{\partial^2 c_k}{\partial x^2} + D_y \frac{\partial^2 c_k}{\partial y^2} + S_k \qquad k=1, 2, \ldots n \qquad (3)$$

where:

c_k is the concentration for the k^{th} pollutant;
u and v are the water velocities in the x and y directions (m/s);
D_x and D_y are the dispersion coefficients in the x and y directions (m^2/s);
S_k is the source term describing the biochemical reactions.

As an example of application, the risk of coastal pollution is considered from a point source, emitting different pollutants, such as nutrients, coliform bacteria and heavy metals. Different case studies have been analysed in the Mediterranean, aiming to establish an optimum design for a submarine outfall discharging wastewater into the sea. The problem is to find the best position for the outfall, so that pollution impacts along the shoreline are kept below the concentration values fixed by the guidelines [7].

Case studies have been developed in the Aegean and Ionian Seas, which form part of the Eastern Mediterranean. Uncertainties exist due to wind-generated currents, which vary randomly both in space and time. As shown in Figure 4, the contour lines of equal impact probability may be computed for a submarine outfall on the Greek island of Rhodes, by tracking a large number of particles, which simulate the wastewater discharge at the outfall's mouth. For every time series of currents, recorded using a submerged current meter, and which corresponds to a certain prevailing wind, regions where pollution exceeds the standards may be obtained. For example, if we know the impact probabilities shown in Figure 4 and the E-coli bacteria concentration at the source to be equal 10^3 per 100 ml, then areas having impact probabilities greater than 0.1 are at risk, because the bacteria concentration is greater than 0.1×10^3 =greater than 100 per 100 ml, which is the maximum allowed E-Coli concentration for bathing waters. By repeating the simulations for different winds, a position for the outfall may be found that results in an acceptable level of risk of bacterial contamination along the shoreline.

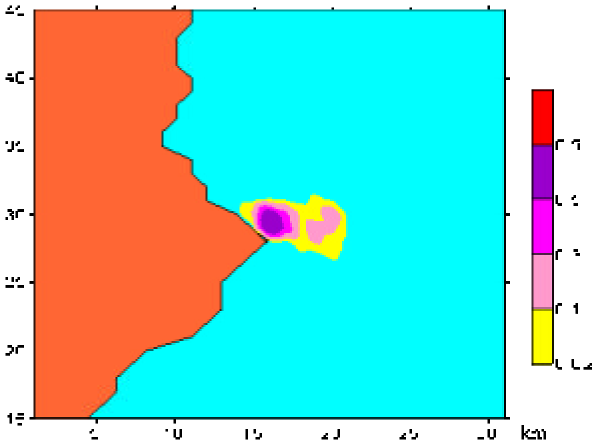


Figure 4. Contour lines of impact probabilities from a submarine outfall

4. Fuzzy Modelling

Let us now assume that only limited information is available on the input pollutant loads (boundary conditions) and the value of the dispersion coefficients. This type of uncertainty can be taken into account by considering both input loads and dispersion coefficients as fuzzy numbers. This implies that unknown pollutant concentrations at any point x, and at any time t, will also behave as fuzzy numbers. They will still follow the advective dispersion partial differential equations, which are written as:

$$\frac{\partial \tilde{c}_k}{\partial t} + u\frac{\partial \tilde{c}_k}{\partial x} + v\frac{\partial \tilde{c}_k}{\partial y} = \tilde{D}_x \frac{\partial^2 \tilde{c}_k}{\partial x^2} + \tilde{D}_y \frac{\partial^2 \tilde{c}_k}{\partial y^2} + S_k \qquad k = 1, 2, \ldots n \qquad (4)$$

where the symbol $\sim$ is used to denote a fuzzy variable.

Although derivatives of fuzzy variables exist, there is no unique solution to equation (4), because fuzzy numbers take different values at different levels of confidence (Appendix). If for every confidence level h, we are looking only for the lower and upper limiting values of the unknown fuzzy variables, then the non-uniqueness problem may be resolved as follows. Every fuzzy number $\tilde{X}$ may be represented by a discrete set of h-level cuts $\overline{X}(h)$ ([12], [13], [22], [23], [21], Appendix). These are ordinary intervals that for the fuzzy variables C and D should also follow the advective dispersion equations written as:

$$\frac{\partial \overline{c}_{k(h)}}{\partial t} + u\frac{\partial \overline{c}_k(h)}{\partial x} + v\frac{\partial \overline{c}_{k(h)}}{\partial y} = \overline{D}_x(h)\frac{\partial^2 \overline{c}_{k(h)}}{\partial x^2} + \overline{D}_y \frac{\partial^2 \overline{c}_k(h)}{\partial y^2} + S_k$$

$$k = 1, 2, \ldots n \qquad (5)$$

where the symbol $-$ is used to denote an ordinary interval.

Applying finite differences or finite elements to equation (5), a system of interval equations needs to be solved. This is difficult from a mathematical point of view, and has stimulated a lot of interest, because whatever possible technique is used, only enclosures for the range of the output function can be produced [5], [6], [11], [14], [17]. Finding the best possible enclosure for an unknown interval function, which is defined as the "hull" of the solution, is a fundamental problem of interval analysis. It should be treated with care, as the solution accuracy depends on the shape of the interval function [18].

Shafike [19] has used the finite element method to simulate a groundwater flow model with fuzzy coefficients. The algebraic system of interval equations was solved with an iterative algorithm [14]. Dou et al. [4] applied the fuzzy set theory to a steady-state groundwater flow model with fuzzy parameters combined with the finite difference method. A non-linear optimisation algorithm was used in order to apply the extension principle for the solution of groundwater flow equations with fuzzy numbers as coefficients for the hydraulic heads. Because of the non-uniqueness of the maximum and minimum values, this may lead to great inaccuracies.

Ganoulis et al. [8], [9], [10] used fuzzy arithmetic to simulate imprecise relations in ecological risk assessment and management. For the solution of the algebraic system of equations with fuzzy coefficients, direct interval operations were employed, instead of the iterative methods or non-linear optimisation techniques used in previous studies.

5. Fuzzy Numerical Simulation

Finite differences and finite elements have been used to produce numerical enclosures of the solution of equation (5) ([7], [15]). Enclosures may fit the exact range of the unknown function better, depending on two main factors:

1. the accuracy of the numerical algorithm, and
2. the correct application of the algebra of fuzzy arithmetic (Appendix).

The Eulerian-Lagrangian scheme, based on the characteristics method, has also been used. In this method the numerical integration of the parabolic part of equation (5) is conducted on the characteristics lines of the equation.

Equation (5) is split into two parts:

1. the hyperbolic part, which in one dimension is written as $dx/dt = u$, and
2. the parabolic part that takes the following form:

$$d\overline{C}(h)/dt = \overline{D}(h)(d^2\overline{C}(h))/dx^2) \tag{6}$$

Using finite differences, a large number of particles are moved over a stationary grid to track the advection equation. At each moment in time $n\Delta t$ the position of each particle is x_p and its concentration is C_p. At the next moment in time $(n+1)\Delta t$ the new position of the particle becomes ${x_p}^{n+1} = {x_p}^{n} + u\Delta t$, and the change of concentration at the nodal points, due to the dispersion part (6), is computed by an explicit finite difference algorithm as follows:

$$\overline{C}_i^{n+1}(h) = \overline{C}_i^{n}(h) + \Delta t\ \overline{D}(h)\,(\overline{C}_{i-1}^{n}(h) - 2\overline{C}_i^{n}(h) + \overline{C}_{i+1}^{n}(h)) \tag{7}$$

The new particle concentrations are evaluated by adding the change due to the diffusive part (7). Then, the new nodal concentrations are calculated by accounting the corresponding number of particle concentrations.

In order to further validate the accuracy of the proposed methodology, another unsteady, one-dimensional case was studied, for which the analytical solution was known [15]. This is the case of the one-dimensional convective dispersion equation (5) with a supplementary term at the right side representing a linear concentration decay of the form $-\tilde{k}\,c$, where $\tilde{k}$ is a fuzzy decay coefficient. The analytical solution has the following form:

$$c = \frac{c_0}{2}\left[e^{T_1}[1 - \text{erf}(S_1)] + e^{T_2}[1 - \text{erf}(S_2)]\right] \tag{9}$$

with

$$F = \sqrt{u^2 + 4kD_x}\ ,\ S_1 = \frac{x - Ft}{\sqrt{4D_x t}}\ ,\ S_2 = \frac{x + Ft}{\sqrt{4D_x t}}\ ,\ T_1 = \frac{u - F}{2D_x}x\ ,\ T_2 = \frac{u + F}{2D_x}x$$

Taking $\tilde{k}$ as a triangular fuzzy number ($1\cdot10^{-5}$, $3\cdot10^{-5}$, $5\cdot10^{-5}$), and using interval representations of fuzzy numbers for h=0, 0.25, 0.5, 0.75, comparison between numerical simulation and analytical solution shown in Figures 5 and 6 indicate that the proposed methodology is very accurate in unsteady water pollution fuzzy simulation.

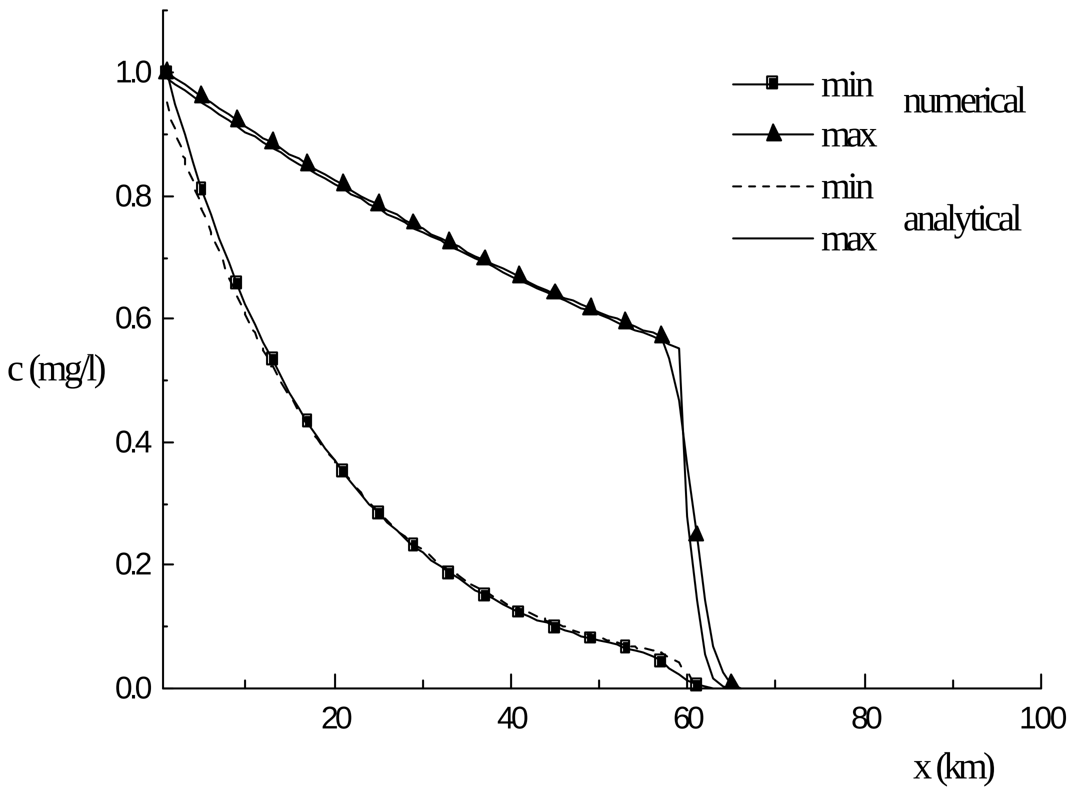
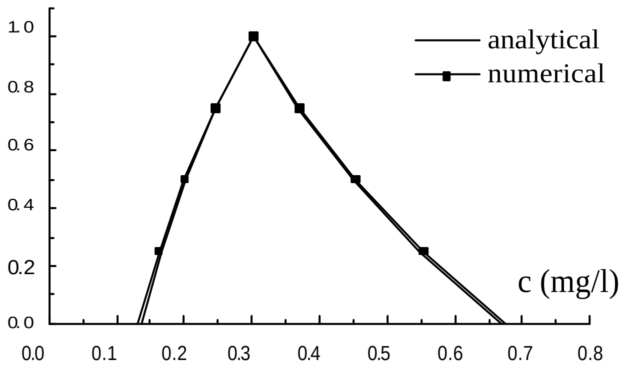
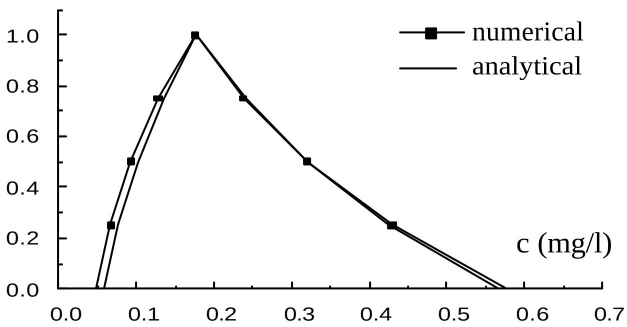


Figure 5. Variation of interval concentrations $\bar{c}$ (h) for h=0, t=16h: comparison between numerical and analytical solutions.



(a)



(b)

Figure 6: Fuzzy concentrations $\tilde{C}$: (a) at x=40 km and (b) at x=57 km, t=16h.

In real geophysical flows, as in coastal circulation in the Bay of Thermaikos, a finite element grid can better describe irregular boundaries. The characteristic Galerkin numerical algorithm was applied for fuzzy simulation of coastal water quality. Velocities of currents were computed using a two-dimensional coastal circulation model. Numerical simulation in the form of fuzzy numbers of output concentration of different pollutants at different locations within the bay incorporates imprecision, due to limited information on land-based pollutant sources and the value of the dispersion coefficients [16].

The finite element grid covering the Bay of Thermaikos and the location of land-based pollutant sources are shown in Figure 7. The model [16] contains ten different pollutants, namely: chlorophyll-a, coliforms, organic nitrogen, ammonia nitrogen, nitrite nitrogen, nitrate nitrogen, organic and inorganic phosphorus, BOD, and dissolved oxygen shortage (or deficit). Results of fuzzy computational modelling are shown in Figure 8 for phytoplankton concentrations at node numbered 200. Knowing the phytoplankton concentration values allowed in national specifications, the risk of eutrophication can be evaluated using the fuzzy risk index.

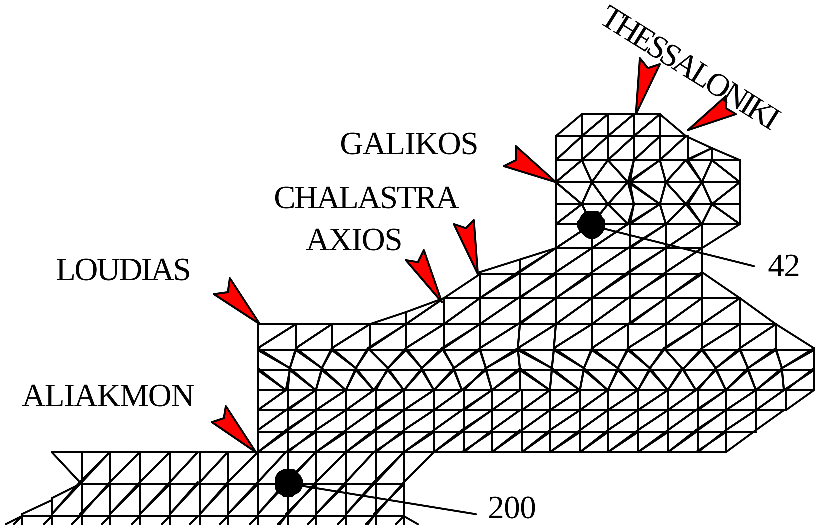


Figure 7. Finite element grid covering the Bay of Thermaikos

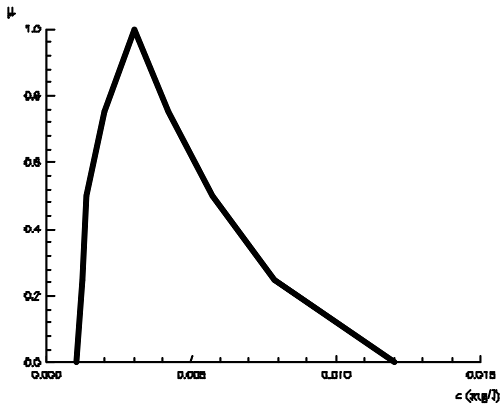


Figure 8. Fuzzy phytoplankton concentration at node 200 .

6. CONCLUSIONS

When limited information or only a few data are available on model parameters and boundary conditions, fuzzy modelling can be used in order to propagate uncertainties and quantify the environmental risk of water systems. Land-based pollutant loads and values of dispersion coefficient may be considered as fuzzy numbers. Uncertainty in output variables as pollutant concentrations can be calculated using fuzzy modelling. For water pollution problems, a fuzzy risk and reliability analysis may be performed; and in combination with fuzzy modelling can be applied in real situations in order to estimate the risk of environmental water pollution.

APPENDIX : FUZZY NUMBERS AND FUZZY ARITHMETIC

A fuzzy number $\tilde{X}$ may be formally defined as a set of ordered pairs

$$\tilde{X} = \{(x, \mu_{\tilde{X}}(x)) : x \in R;\ \mu_{\tilde{X}}(x) \in [0, 1]\} \qquad (A.1)$$

where x is a particular value of $\tilde{X}$ and $\mu_{\tilde{X}}(x)$ represents its membership function. Values of the membership function are located in the closed interval [0,1]. The closer $\mu_{\tilde{X}}(x)$ is to 1, the more "certain" one is about the value of x. A fuzzy number $\tilde{X}$ is normal and convex when its membership function takes one maximum value equal to 1 and always increases to the left of the peak, and decreases to the right.

The simplest type of fuzzy number is the triangular, which is one having linear membership functions on either side of the peak. A fuzzy triangular number can be characterised by three real numbers: two values of x i.e., x_1, x_2, where the membership function reaches zero, and one value x_3, where it reaches a value of 1.

A triangular fuzzy number (TFN) may be described by the values of x at points x_1, x_2 and, i.e. $\tilde{X} = (x_1, x_2, x_3)$

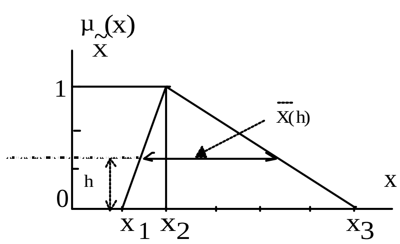


Figure A.1. A triangular fuzzy number $\tilde{X}=(x_1, x_2, x_3)$.

The h-level set of a fuzzy number $\tilde{X}$ (Fig. A.1) is the ordinary set or interval $\overline{X}(h)$, defined as

$$\overline{X}(h)=\{x: \mu_{\tilde{X}}(x) \geq h)\} \tag{A.2}$$

Let us consider two triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ given by the triplets $\tilde{A}=(a_1, a_2, a_3)$ and $\tilde{B}=(b_1, b_2, b_3)$. We have

i) *Addition:* $\tilde{A}+\tilde{B} = (a_1+b_1, a_2+b_2, a_3+b_3)$

ii) *Subtraction:* $\tilde{A}-\tilde{B} = (a_1-b_3, a_2-b_2, a_3-b_1)$

The multiplication or division of two fuzzy numbers does not always produce a fuzzy number. These operations can be defined as follows:

iii) *Multiplication*

$$\tilde{A}*\tilde{B}=[\min(a_1*b_1, a_1*b_3, a_3*b_1, a_3*b_3), a_2*b_2, \max(a_1*b_1, a_1*b_3, a_3*b_1, a_3*b_3)]$$

iv) *Division*

$$\tilde{A}/\tilde{B}=[\min(\frac{a_1}{b_1}, \frac{a_1}{b_3}, \frac{a_3}{b_1}, \frac{a_3}{b_3}), \frac{a_2}{b_2}, \max(\frac{a_1}{b_1}, \frac{a_1}{b_3}, \frac{a_3}{b_1}, \frac{a_3}{b_3})]$$

<u>Algebraic Properties of Fuzzy Numbers</u>

Let us assume that $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are fuzzy numbers. The following laws hold:

Cummutativity	$\tilde{A}+\tilde{B}=\tilde{B}+\tilde{A}$	$\tilde{A}\,\tilde{B}=\tilde{B}\,\tilde{A}$
Associativity	$(\tilde{A}+\tilde{B})\pm\tilde{C} = \tilde{A}+(\tilde{B}\pm\tilde{C})$	$(\tilde{A}\,\tilde{B})\tilde{C}=\tilde{A}(\tilde{B}\,\tilde{C})$

However, subdistributivity and subcancellation are not always valid. In particular we have

Subdistribution

$\tilde{A}(\tilde{B}\pm\tilde{C}) \subseteq \tilde{A}\,\tilde{B} \pm \tilde{A}\,\tilde{C}$ $\qquad$ $(\tilde{A}\pm\tilde{B})\tilde{C} \subseteq \tilde{A}\,\tilde{C} \pm \tilde{B}\,\tilde{C}$

Subcancellation

$\tilde{A}-\tilde{B} \subseteq (\tilde{A}+\tilde{C})-(\tilde{B}+\tilde{C})$ $\qquad$ $\tilde{A}/\tilde{B} \subseteq \tilde{A}\,\tilde{C}/\tilde{B}\,\tilde{C}$

Overestimation usually occurs because of the failure of the distributive and cancellation laws.

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